# Weiss-approach to pair of coupled non-linear reaction-diffusion equations

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March 1, 2003

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**Abstract**

We consider a pair of coupled non-linear partial differential equations describing a biochemical model system. The Weiss-algorithm for the Painlevé test, that has been succesfully used in mathemathical physics for the KdV-equation, Burgers equation, the sine-Gordon equation etc., is applied, and we find that the system possesses only the "conditional" Painlevé property. We use the outcome of the analysis to construct an auto-Bäcklund transformation, and we find a variety of one and two-parameter families of special solutions.

# 1 Introduction

Although the deeper reasons are still not completely understood, it is clear today that the Painlevé property [1] of (systems of) ordinary or partial differential equations is closely related to the concept of integrability. The original approach of Ablowitz, Ramani and Segur [2], led to the conjecture that a non-linear partial differential equation (PDE) is integrable if every exact reduction to an ordinary differential equation (ODE) has the Painlevé property, i.e. has no other movable singularities than poles. This version of the Painlevé conjecture is well-suited to confirm already known integrability properties of a PDE, but is less helpfull when it comes to the act of finding new integrable systems, due to the problems of actually finding all the exact reductions. In this respect the later approach of Weiss, Tabor and Carnevale [3] seems to be much more usefull. In the simplest version of this approach a PDE is conjectured to be integrable if its solutions are singlevalued about movable "singularity-manifolds":

$$\phi(z_1, z_2, ..., z_n) = 0, \tag{1.1}$$

where $\phi$ is an arbitrary ("movable") analytic function. In other words a solution $x(z_i)$ to the PDE in question should have a Laurent-like expansion about the movable singular manifold (1.1):

$$x(z_i) = [\phi(z_i)]^\rho \sum_{j=0}^{\infty} [\alpha_j(z_i)][\phi(z_i)]^j, \tag{1.2}$$

where $\rho$ is a negative integer. If the number of arbitrary functions in the expansion (1.2) equals the order of the PDE, we have reason to believe that (1.2) is the generic expansion about $\phi = 0$. If this is not the case we may have lost something and the system does not have the full Painlevé property. Further details can be found in Refs. 1,3.

This Weiss-approach to the Painlevé property has been used to analyse a long list of the PDE's of mathematical physics like the KdV-equation, Burgers equation, sine-Gordon equation, modified KdV-equation, Boussinesq equation,... [3,4]. The method is of course also applicable to systems of coupled non-linear PDE's like the coupled KdV-equations (Hirota-Satsuma system) [5]:

$$x_t = \frac{1}{2}x_{rrr} + 3xx_r - 6yy_r,$$

$$y_t = -y_{rrr} - 3xy_r,$$

or the modified sine-Gordon system [5]:

$$x_t + \frac{1}{2}xy + \frac{\lambda}{2}\frac{x}{y} = 0,$$
$$y_r + \frac{1}{2}xy + \frac{1}{2\lambda}\frac{y}{x} = 0.$$

Systems of coupled non-linear PDE's also arise naturally in (mathematical) biology and chemistry, for instance in models describing reactions and diffusion of different components in a medium. In this paper we will consider the following system of PDE's [6-10]:

$$\begin{aligned} x_t &= x^2 y - Bx + Kx_{rr}, \\ y_t &= -x^2 y + Bx + Ky_{rr}, \end{aligned} \tag{1.3}$$

where $B$ and $K$ are constants. These equations are supposed to describe the evolution of concentrations $x$ and $y$ of 2 chemical components. Obviously $K$ is a diffusion coefficient and the terms $x^2y - Bx$ represent the irreversible reactions. Such processes are certainly not expected to be integrable and we will indeed see that (1.3) does not have the full Painlevé property (Sec. 2). We note in passing that the system is closely related to the so-called Brusselator [6] describing the scheme of reactions:

$$A \to X, \quad B + X \to Y + C,$$
$$2X + Y \to 3X, \quad X \to D,$$

where $A, B, C, D, X$ and $Y$ are different kinds of molecules.

The system of equations (1.3) has been investigated by several authors [6-10]. The steady state solutions ($x_t = y_t = 0$) were analysed in Ref. 7 under the boundary condition $x + y = const$. In that case Eqs. (1.3) trivially separate and are completely solvable in terms of elliptic functions. In Refs. 8 and 9 a system like (1.3), but with 2 different diffusion coefficients, was investigated. It was assumed that one of the components diffuses very rapidly so that one of the diffusion coefficients could be taken to be infinite. The system was then reduced to a one-variable non-linear PDE and some travelling wave solutions were found. Finally we mention that a one-parameter family of travelling wave solutions for the coupled system (1.3) was found in Ref. 8 under the boundary condition $x + y = const$.

The paper is organized as follows: In section 2 we carry out the Painlevé analysis of the system (1.3) using the Weiss-approach. We consider expansions of $x$ and $y$ about a movable singular manifold and find that the system possesses only the so-called conditional Painlevé property. In section 3 we truncate the expansions of section 2 at the most singular term, which allows us to construct a two-parameter family of special solutions in terms of trigonometric functions. In section 4 we truncate the expansions of section 2 at the "constant" term. This leads to an auto-Bäcklund transformation between 2 pairs of solutions. We then construct some more one and two-parameter families of special solutions, and in section 5 we finally give our conclusions and we outline how more general families of solutions can be "chased".

## 2 Painlevé test

In this section we perform the Painlevé analysis of the system of coupled non-linear PDE's (1.3) with $B$ and $K$ as arbitrary positive constants (the positivity of $B$ and $K$ is of course not mandatory, at least not from a mathematical point of view).

The first step in the Weiss-algorithm [3] is to look for the dominant behaviour about a movable singular manifold $\phi(r,t)=0$. Thus we write:

$$x = U_o(r,t)[\phi(r,t)]^\rho \equiv U_o\phi^\rho; \quad \rho < 0, \tag{2.1}$$

$$y = V_o(r,t)[\phi(r,t)]^\sigma \equiv V_o\phi^\sigma; \quad \sigma < 0, \tag{2.2}$$

and balances the most singular terms after insertion in (1.3). This leads to the unique solution:

$$\rho = \sigma = -1, \quad U_o = -V_o, \tag{2.3}$$

with:

$$U_o = \pm\sqrt{2K}\phi_r. \tag{2.4}$$

In the following we only consider the case of +-sign in (2.4). The other possibility corresponds to an overall change of sign of $x$ and $y$. Clearly if $(x,y)$ solves (1.3) also $(-x,-y)$ is a solution.

The next step is to look for the "resonances" [3], i.e. the orders in the expansions where arbitrary functions may appear. Keeping (2.3) and (2.4) in mind we write:

$$x = \sqrt{2K}\phi_r\phi^{-1} + s_1\phi^{l-1}, \tag{2.5}$$

$$y = -\sqrt{2K}\phi_r\phi^{-1} + s_2\phi^{l-1}, \tag{2.6}$$

where $l$ is a non-negative integer and $s_i = s_i(r,t); i = 1,2$. After insertion in (1.3) and balancing of the most singular terms we get the matrix-equation:

$$\begin{pmatrix} (l-1)(l-2)-4 & 2 \\ 4 & (l-1)(l-2)-2 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \equiv \mathcal{A}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.7}$$

with:

$$\det\mathcal{A} = (l-1)(l-2)[(l-1)(l-2)-6]. \tag{2.8}$$

For a given root of the polynomium (2.8) there are now as many arbitrary functions $s_i$ as the multiplicity of that particular root. Solving the equation

$\det\mathcal{A} = 0$ we find the roots $(-1, 1, 2, 4)$. The root $l = -1$ corresponds to the arbitraryness of the location of the singular manifold $\phi$, while the roots $l = 1$, $l = 2$ and $l = 4$ are supposed to correspond to arbitrary functions at the orders $\phi^0$, $\phi^1$ and $\phi^3$ in the expansions of $x$ and $y$ about $\phi = 0$. Note that the number of roots equals the order of the system (1.3) so until now everything works fine.

The third and final step in the Weiss-algorithm [3] is to expand out to the highest resonance to make sure that no inconsistencies arise, i.e. we write:

$$x = \sqrt{2K}\phi_r\phi^{-1} + \sum_{j=0}^{3}\alpha_j\phi^j; \quad \alpha_j = \alpha_j(r,t), \tag{2.9}$$

$$y = -\sqrt{2K}\phi_r\phi^{-1} + \sum_{j=0}^{3}\beta_j\phi^j; \quad \beta_j = \beta_j(r,t). \tag{2.10}$$

These expressions are inserted into (1.3) and we then balance the terms order by order in $\phi$. It was demonstrated by Kruskal [11] that some simplifications arise in this process if one formally solves the equation $\phi(r,t) = 0$ for (say) $r$ and then writes:

$$\phi(r,t) = r + \psi(t), \quad \alpha_j = \alpha_j(t), \quad \beta_j = \beta_j(t), \tag{2.11}$$

where $\psi$ is an arbitrary function. For our purposes of constructing explicit special solutions (sections 3,4) it is however necessary to use the general expansions (2.9) and (2.10).

At the various orders we now get:
$\phi^{-2}$:

$$\sqrt{2K}(2\alpha_0 - \beta_0) = \frac{\phi_t}{\phi_r} - 3K\frac{\phi_{rr}}{\phi_r}, \tag{2.12}$$

so here we get an arbitrary function (say) $\alpha_0$, as expected from the analysis of the resonances.
$\phi^{-1}$:

$$\sqrt{2K}(2\alpha_1 - \beta_1) = \frac{\alpha_0(2\beta_0 - \alpha_0)}{\phi_r} - \frac{\phi_{rt}}{\phi_r^2} - \frac{B}{\phi_r} + K\frac{\phi_{rrr}}{\phi_r^2}, \tag{2.13}$$

with another arbitrary function (say) $\alpha_1$.

$\phi^0$:

$$\left\{ \begin{array}{l} \alpha_2 = F_1(\alpha_0, \alpha_1, \phi; B, K) \\ \beta_2 = F_2(\alpha_0, \alpha_1, \phi; B, K) \end{array} \right. , \tag{2.14}$$

where $F_1$ and $F_2$ are certain complicated expressions in the arbitrary functions $\phi$, $\alpha_0$, $\alpha_1$ and their derivatives, as well as in the "chemical" constants $B$ and $K$. For convenience they are listed in the appendix.

$\phi^1$:

$$\left\{ \begin{array}{l} \alpha_3 + \beta_3 = G_1(\alpha_0, \alpha_1, \phi; B, K) \\ \alpha_3 + \beta_3 = G_2(\alpha_0, \alpha_1, \phi; B, K) \end{array} \right. . \tag{2.15}$$

There is now an arbitrary function at this order if and only if the 2 right hand sides are equal. From the explicit expressions for $G_1$ and $G_2$ given in the appendix it follows that this is not so (this is actually most easily seen by using the Kruskal *Ansatz* (2.11) and by keeping in mind the arbitraryness of (say) $\alpha_0$ and $\alpha_1$). It follows that (say) $\alpha_3$ is arbitrary only if the singularity manifold $\phi$ satisfies a certain constraint. Therefore, the system (1.3) does not have the full Painlevé property but only the "conditional" one [12].

# 3 Truncation and special solutions

In this and the following section we will extensively use the expressions among the alpha's and beta's obtained in section 2. In this section we look for special solutions to (1.3) obtained by truncation of the expansions (2.9) and (2.10) at the singular terms. Taking $\alpha_i = \beta_i = 0; i \geq 0$ we find:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2K}\frac{\phi_r}{\phi}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.1}$$

This can however only be a solution to (1.3) provided equations (2.12)-(2.15) are fulfilled. It is easily seen that quations (2.14) and (2.15) are trivially satisfied whereas (2.12) and (2.13) give the compatibility conditions:

$$\frac{\phi_t}{\phi_r} - 3K\frac{\phi_{rr}}{\phi_r} = 0, \tag{3.2}$$

$$\frac{B}{\phi_r} + \frac{\phi_{rt}}{\phi_r^2} - K\frac{\phi_{rrr}}{\phi_r^2} = 0. \tag{3.3}$$

These 2 equations are integrated and consistently solved by:

$$\phi(r,t) = e^{-\frac{3}{2}Bt}\left(c_1\cos(\sqrt{\frac{B}{2K}}r) + c_2\sin(\sqrt{\frac{B}{2K}}r)\right) + c_3, \tag{3.4}$$

where $c_1, c_2, c_3$ are arbitrary constants. From (3.1) we then get:

$$x(r,t) = -y(r,t) = \frac{\sqrt{B}e^{-\frac{3}{2}Bt}\left(L_1\cos(\sqrt{\frac{B}{2K}}r) - L_2\sin(\sqrt{\frac{B}{2K}}r)\right)}{e^{-\frac{3}{2}Bt}\left(L_2\cos(\sqrt{\frac{B}{2K}}r) + L_1\sin(\sqrt{\frac{B}{2K}}r)\right) + 1}, \tag{3.5}$$

representing a two-parameter family ($L_1$ and $L_2$ being 2 arbitrary constants to be determined by the initial/boundary conditions) of solutions to the system (1.3).

# 4 Auto-Bäcklund transformation and special solutions

In this section we truncate the expansions (2.9) and (2.10) at order $\phi^0$, i.e. we take $\alpha_i = \beta_i = 0; i \geq 1$, and look for solutions in the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2K}\frac{\phi_r}{\phi}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}. \tag{4.1}$$

As in section 3 this can of course only be a solution to (1.3) provided (2.12)-(2.15) are fulfilled. Equations (2.12) and (2.13) read:

$$\sqrt{2K}(2\alpha_0 - \beta_0)\phi_r = \phi_t - 3K\phi_{rr}, \tag{4.2}$$

$$\alpha_0(2\beta_0 - \alpha_0) = \frac{\phi_{rt}}{\phi_r} + B - K\frac{\phi_{rrr}}{\phi_r}. \tag{4.3}$$

After using (6.1) and (6.2) from the appendix, equation (2.14) leads to:

$$\begin{aligned} \alpha_{0t} &= \alpha_0^2\beta_0 - B\alpha_0 + K\alpha_{0rr}, \\ \beta_{0t} &= -\alpha_0^2\beta_0 + B\alpha_0 + K\beta_{0rr}, \end{aligned} \tag{4.4}$$

while (2.15) is trivially fulfilled. (4.2)-(4.4) represents an overdetermined system of equations to be solved for $\alpha_0, \beta_0$ and $\phi$. This system is extremely complicated in the general case but fortunately it is not so difficult to find special solutions. Note also that (4.4) has the same form as the original system (1.3), so if we can find solutions $(\alpha_0, \beta_0, \phi)$ to (4.2)-(4.4) it follows that (4.1) is an auto-Bäcklund transformation between the 2 pairs of solutions $(x, y)$ and $(\alpha_0, \beta_0)$ to equation (1.3). It means that if we find one pair of solutions we can in principle always generate new ones.

Special solutions to the overdetermined system (4.2)-(4.4) can conveniently be parametrized by the 2 functions [12]:

$$C \equiv \frac{\phi_t}{\phi_r}, \qquad V \equiv \frac{\phi_{rr}}{\phi_r}. \tag{4.5}$$

For simplicity we will now restrict ourselves by considering only constant $C$ and $V$. It turns out that two different types of solutions are possible corresponding to $V = 0$ and $V \neq 0$. In the case that $V = 0$ and $C \equiv C_0$ is an arbitrary constant we find:

$$\phi(r, t) = c_1(r + C_0 t) + c_2, \tag{4.6}$$

where $c_1, c_2$ are arbitrary constants. Equations (4.2)-(4.4) lead to:

$$\begin{aligned} 2\alpha_0 - \beta_0 &= \tfrac{C_0}{\sqrt{2K}}, \\ \alpha_0(2\beta_0 - \alpha_0) &= B, \\ \alpha_0^2\beta_0 - B\alpha_0 &= 0, \end{aligned} \tag{4.7}$$

that are solved by:

$$\alpha_0 = \beta_0 = \pm\sqrt{B}, \qquad C_0 = \pm\sqrt{2KB}. \tag{4.8}$$

Using (4.1) we then get:

$$x(r, t) = -y(r, t) \pm 2\sqrt{B} = \frac{\sqrt{2K}}{(r \pm \sqrt{2KB}t) + L_1} \pm \sqrt{B}, \tag{4.9}$$

representing a one-parameter family ($L_1$ being an arbitrary constant) of solutions to (1.3).

In the case that both $V \equiv V_0$ and $C \equiv C_0$ are arbitrary constants ($V_0 \neq 0$) we find instead:

$$\phi(r, t) = c_1 e^{V_0(r + C_0 t)} + c_2, \tag{4.10}$$

where $c_1, c_2$ are arbitrary constants. In this case equations (4.2)-(4.4) lead to:

$$\begin{aligned} \sqrt{2K}(2\alpha_0 - \beta_0) &= C_0 - 3KV_0, \\ \alpha_0(2\beta_0 - \alpha_0) &= V_0C_0 + B - KV_0^2, \\ \alpha_0^2\beta_0 - B\alpha_0 &= 0, \end{aligned} \tag{4.11}$$

Solving (4.11) for $(\alpha_0, \beta_0, C_0)$ in terms of $(B, K, V_0)$ leads to the 3 possibilities:

$$(\alpha_0, \beta_0, C_0) = \left(-\sqrt{2K}V_0, \frac{-B}{\sqrt{2K}V_0}, \frac{B}{V_0} - KV_0\right), \tag{4.12}$$

and:

$$(\alpha_0, \beta_0, C_0) = \left(\frac{-\sqrt{2K}V_0 \pm \sqrt{W_0}}{2}, \frac{2B}{-\sqrt{2K}V_0 \pm \sqrt{W_0}}, \frac{W_0\sqrt{2K} \mp KV_0\sqrt{W_0}}{-\sqrt{2K}V_0 \pm \sqrt{W_0}}\right), \tag{4.13}$$

where $W_0 \equiv 2B + KV_0^2$. From (4.1) we finally get:

$$x(r,t) = -y(r,t) + \alpha_0 + \beta_0 = \frac{\sqrt{2K}V_0 e^{V_0(r+C_0t)}}{e^{V_0(r+C_0t)} + L_1} + \alpha_0, \tag{4.14}$$

with $(\alpha_0, \beta_0, C_0)$ given by one of (4.12), (4.13). This equation finally represents a two-parameter family ($L_1$ and $V_0$ being arbitrary constants) of solutions to (1.3).

# 5 Conclusion

In conclusion we have studied the pair of reaction-diffusion equations (1.3) using the Weiss-algorithm for the Painlevé test. The system was found to possess only the conditional Painlevé property. The results of the analysis however led to various kinds of special solutions obtained via truncations and auto-Bäcklund transformations.

The special solutions we have constructed (3.5), (4.9) and (4.14) are all of the form $x(r,t) + y(r,t) = const$. Note that addition of the two equations in (3.1) leads to:

$$(x+y)_t = K(x+y)_{rr}, \tag{5.1}$$

which is just the ordinary one-variable diffusion equation with the well-known general solution (see for instance reference 13):

$$(x+y)(r,t) = \frac{1}{2\sqrt{\pi K}} \int_{-\infty}^{\infty} \frac{F(\xi)}{\sqrt{t}} e^{-\frac{(r-\xi)^2}{4Kt}} d\xi, \tag{5.2}$$

where:

$$(x+y)(r,0) = F(r). \tag{5.3}$$

All our solutions therefore correspond to the boundary condition $F(r) = const.$, and could in principle have been obtained by applying the Weiss-algorithm to a one-variable non-linear reaction-diffusion equation obtained from (3.1) by separating $x$ and $y = -x + const.$ from the beginning.

The advantage of our more general approach where (in the notation of (5.2)) $F(r)$ is not fixed from the beginning , is of course that we can now use the results of sections 2-4 to look for more general families of solutions with non-constant $F(r)$: We can either return to equation (4.5) and look for solutions to (4.2)-(4.4) with non-constant $V$ and/or $C$, or we can continue along the road of sections 2 and 3 and look at expansions truncated at higher orders in $\phi$. That is however out of the scope of this paper.

**Acknowledgements**: I would like to thank J.B.Pedersen for a discussion on general aspects of diffusion and reactions.

# 6 Appendix

In this appendix we list the functions $(F_1, F_2)$ appearing in (2.14) and the functions $(G_1, G_2)$ appearing in (2.15).

The functions $F_1$ and $F_2$ are given by:

$$\begin{aligned} 4K\phi_r^2 F_1 &= \beta_{0t} - \beta_1\phi_t + \alpha_0^2\beta_0 - B\alpha_0 - K\beta_{0rr} - 2K\beta_{1r}\phi_r \\ &- K\beta_1\phi_{rr} + 2\sqrt{2K}\phi_r(\alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_0\alpha_1), \end{aligned} \tag{6.1}$$

$$\begin{aligned} 4K\phi_r^2 F_2 &= (2\alpha_0 + \beta_0)_t + (2\alpha_1 + \beta_1)\phi_t - K(2\alpha_0 + \beta_0)_{rr} \\ &- 2K(2\alpha_1 + \beta_1)_r\phi_r + B\alpha_0 - \alpha_0^2\beta_0 - K(2\alpha_1 + \beta_1)\phi_{rr} \\ &- 2K\sqrt{2K}\phi_r(\alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_0\alpha_1). \end{aligned} \tag{6.2}$$

Using (2.12) and (2.13) it is straightforward to express $F_1$ and $F_2$ in terms of (say) the arbitrary functions $(\alpha_0, \alpha_1, \phi)$ and their derivatives as well as the "chemical" constants $(B, K)$.

The functions $G_1$ and $G_2$ are given by:

$$\begin{aligned} 4K\phi_r^2 G_1 &= 2\alpha_{1t} + 4\alpha_2\phi_t + 2\sqrt{2K}\phi_r(2\alpha_0\alpha_2 + \alpha_1^2) + 2B\alpha_1 \\ &- 4\beta_0(\sqrt{2K}\phi_r\alpha_2 + \alpha_0\alpha_1) - 2\beta_1(2\sqrt{2K}\phi_r\alpha_1 + \alpha_0^2) \\ &- 4\sqrt{2K}\beta_2\phi_r\alpha_0 - 2K(\alpha_{1rr} + 4\alpha_{2r}\phi_r + 2\alpha_2\phi_{rr}), \end{aligned} \tag{6.3}$$

$$\begin{aligned} 4K\phi_r^2 G_2 &= \beta_{1t} + 2\beta_2\phi_t - \sqrt{2K}\phi_r(2\alpha_0\alpha_2 + \alpha_1^2) - B\alpha_1 \\ &+ 2\beta_0(\sqrt{2K}\phi_r\alpha_2 + \alpha_0\alpha_1) + \beta_1(2\sqrt{2K}\phi_r\alpha_1 + \alpha_0^2) \\ &+ 2\sqrt{2K}\beta_2\phi_r\alpha_0 - K(\beta_{1rr} + 4\beta_{2r}\phi_r + 2\beta_2\phi_{rr}). \end{aligned} \tag{6.4}$$

Using (6.1), (6.2) and (2.12), (2.13) we can again express the right hand sides in terms of (say) the arbitrary functions $(\alpha_0, \alpha_1, \phi)$ and their derivatives as well as the "chemical" constants $(B, K)$. It can then be verified that $G_1 \neq G_2$, so that both $\alpha_3$ and $\beta_3$ are fixed by (2.15), i.e. there is no arbitrary function at this order.

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